

Assignment 3

Exercise 1

The goal of this exercise is to mimic the construction of Brownian motion done in the lectures to construct the Poisson process, which is a much simpler yet important process. Recall that Γ follows a Poisson distribution $P(\lambda)$ of parameter $\lambda > 0$ if $\mathbb{P}(\Gamma = k) = e^{-\lambda}\lambda^k/k!$ for all $k \in \mathbb{N}$. We fix throughout a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here are some simple questions concerning Poisson random variables.

- 1) Let $\Gamma \sim P(\lambda)$ and $\Gamma' \sim P(\lambda')$, for $(\lambda, \lambda') \in (0, +\infty)^2$ be \mathbb{P} -independent. Show that $\Gamma + \Gamma' \sim P(\lambda + \lambda')$.
- 2) Suppose that $\Gamma \sim P(\lambda)$ and $p \in (0, 1)$. Let $(X_i)_{i \in \mathbb{N}^*}$ be \mathbb{P} -i.i.d. random variables \mathbb{P} -independent of Γ with $\mathbb{P}[X_i = 1] = 1 - \mathbb{P}[X_i = 0] = p$, $i \in \mathbb{N}^*$, and define $\Gamma_p := \sum_{i=1}^{\Gamma} X_i$ and $\Gamma_{1-p} := \sum_{i=1}^{\Gamma} (1 - X_i)$. Show that $\Gamma_p \sim P(p\lambda)$, $\Gamma_{1-p} \sim P((1-p)\lambda)$ and that Γ_p and Γ_{1-p} are \mathbb{P} -independent.
- 3) Determine the characteristic function of $P(\lambda)$.

We are now going to construct a continuous-time process $N := (N_t)_{t \in \mathbb{R}_+}$ with values in $\mathbb{N} \cup \{+\infty\}$ satisfying the following properties (N is called a *Poisson process* of rate 1):

- $N_0 = 0$, \mathbb{P} -a.s.;
- $N_t \sim P(t)$, for all $t > 0$;
- N has \mathbb{P} -independent and stationary increments, that is for all $n \in \mathbb{N}^*$ and any $0 \leq t_0 < t_1 < \dots < t_n$, we have that $(N_{t_i} - N_{t_{i-1}} : i \in \{1, \dots, n\})$ are \mathbb{P} -independent and for any $0 \leq s < t$, $N_t - N_s$ and N_{t-s} have the same law;
- $t \mapsto N_t$ is right-continuous and non-decreasing.

The goal now is to construct such a process using a countable collection of \mathbb{P} -i.i.d. Poisson random variables with parameter 1 and an independent countable number of i.i.d. Bernoulli random variables with parameter $1/2$.

- 4) Show iteratively that we can construct a process $(N'_t)_{t \in D_n}$ satisfying the first three properties above where for any $n \in \mathbb{N}$, $D_n := 2^{-n}\mathbb{N}$. Check that $t \mapsto N'_t$ defined on $\cup_{n \in \mathbb{N}} D_n$ is \mathbb{P} -a.s. non-decreasing.
- 5) Now define $N_t := \inf\{N'_s : s > t, s \in \cup_{n \in \mathbb{N}} D_n\}$ for $t \geq 0$. Show that N is a Poisson process.

We now give another construction of a Poisson process. Recall that we say that a random variable τ has an exponential distribution with parameter $\lambda \geq 0$ if its law has a density with respect to Lebesgue measure given by $\lambda e^{-\lambda t} \mathbf{1}_{(0, \infty)}(t)$, $t \in \mathbb{R}$. Let $(\tau_i)_{i \in \mathbb{N}^*}$ be \mathbb{P} -i.i.d. exponentially distributed random variables with parameter 1, and set $N_t := \sup\{k \geq 0 : \tau_1 + \dots + \tau_k \leq t\}$, $t \geq 0$.

- 6) Show that $N_0 = 0$, \mathbb{P} -a.s., and that $t \mapsto N_t$ is right-continuous and non-decreasing.
- 7) Show for any $1 \leq i \leq j$ (by induction on $j - i$ or otherwise) that the law of $\tau_{[i, j]} := \tau_i + \dots + \tau_j$ is has a density with respect to Lebesgue measure given by $t^{j-i} e^{-t} / (j - i)! \mathbf{1}_{(0, \infty)}(t)$ (this is a Gamma distribution).
- 8) By explicit computation show that N is a Poisson process.

1) **Let $k \in \mathbb{N}$, then**

$$\mathbb{P}[\Gamma + \Gamma' = k] = \sum_{\ell=0}^k \mathbb{P}[\Gamma = \ell, \Gamma' = k - \ell] = \sum_{\ell=0}^k e^{-\lambda} \frac{\lambda^\ell}{\ell!} e^{-\lambda'} \frac{(\lambda')^{k-\ell}}{(k-\ell)!} = \frac{e^{-(\lambda+\lambda')}}{k!} \sum_{\ell=0}^k \binom{k}{\ell} \lambda^\ell (\lambda')^{k-\ell} = e^{-(\lambda+\lambda')} \frac{(\lambda + \lambda')^k}{k!},$$

as required (note that we used independence in the third equality).

2) Take $(k, \ell) \in \mathbb{N}^2$ and observe that

$$\begin{aligned} \mathbb{P}[\Gamma_p = k, \Gamma_{1-p} = \ell] &= \mathbb{P}[\Gamma = k + \ell, X_1 + \dots + X_{k+\ell} = k] = \mathbb{P}[\Gamma = k + \ell] \mathbb{P}[X_1 + \dots + X_{k+\ell} = k] \\ &= e^{-\lambda} \frac{\lambda^{k+\ell}}{(k+\ell)!} \binom{k+\ell}{k} p^k (1-p)^\ell \\ &= e^{-\lambda p} \frac{(\lambda p)^k}{k!} e^{-\lambda(1-p)} \frac{(\lambda(1-p))^\ell}{\ell!}. \end{aligned}$$

Here we used that Γ and $X_1 + \dots + X_{k+\ell}$ are \mathbb{P} -independent and that the latter random variable is $B(k+\ell, p)$ distributed. The claim follows.

3) The characteristic function ϕ of $P(\lambda)$ is given by

$$\phi(\theta) = \sum_{k=0}^{\infty} e^{i\theta k} e^{-\lambda} \frac{\lambda^k}{k!} = e^{\lambda(e^{i\theta} - 1)}.$$

4) Let $(\Gamma_i)_{i \in \mathbb{N}^*}$ be i.i.d. $P(1)$ random variables with values in \mathbb{N} . For $t \in D_0 = \mathbb{N}$, we let $N'_t := \Gamma_1 + \dots + \Gamma_t$ (with the convention $N'_0 = 0$). Independence of increments is immediate from the construction, and stationarity and $N'_t \sim P(t)$ are clear from 1). Now suppose we have constructed $(N'_t)_{t \in D_n}$ satisfying the first three properties of the definition and we need to now define the process on $D_{n+1} \setminus D_n$. For $t \in D_{n+1} \setminus D_n$, observe that $t \pm 2^{-(n+1)} \in D_n$. Let $(X_i^t)_{i \in \mathbb{N}^*}$ be i.i.d Bernoulli $1/2$ random variables (independent of everything else in the construction) and set

$$N'_t := N'_{t-2^{-(n+1)}} + \sum_{i=1}^{N'_{t+2^{-(n+1)}} - N'_{t-2^{-(n+1)}}} X_i^t.$$

By part 2) with $p = 1/2$, $(N'_t - N'_{t-2^{-(n+1)}})_{t \in D_{n+1} \setminus D_n}$ are independent and all $P(2^{-(n+1)})$ distributed. The first three properties are then checked to be satisfied using 1). N' is \mathbb{P} -a.s. non-decreasing: indeed, for $s < t \in \cup_{n \in \mathbb{N}} D_n$, say $(s, t) \in D_{n_0}^2$ for some $n_0 \in \mathbb{N}$, $N'_t - N'_s \sim P(t-s)$ so $\mathbb{P}[N'_t - N'_s \geq 0] = 1$. Since $\cup_{n \in \mathbb{N}} D_n$ is countable, the claim follows.

5) First, $0 \leq \mathbb{E}^\mathbb{P}[N_0] \leq \liminf_{n \rightarrow +\infty} \mathbb{E}^\mathbb{P}[N'_{2^{-n}}] = \liminf_{n \rightarrow \infty} 2^{-n} = 0$ by Fatou's lemma, and so $N_0 = 0$, \mathbb{P} -a.s. Since N' is \mathbb{P} -a.s. non-decreasing, for fixed $t \geq 0$ and $(t_k) \subset \cup_{n \in \mathbb{N}} D_n$ with $t_k \downarrow t$, we have $N'_{t_k} \rightarrow N'_t$, \mathbb{P} -a.s. as $k \rightarrow +\infty$. For example by inspecting the characteristic function, we see that therefore $N_t \sim P(t)$. Stationarity and independence of increments follows using an analogous argument. Finally, it is easy to see from the definition that N is right-continuous and non-decreasing.

6) It is in fact clear that $N_0 = 0$. Moreover, N is non-decreasing, since increasing t in the definition of N_t enlarges the set over which the supremum is taken. For right-continuity, fix $t \geq 0$. If $N_t = \infty$ (which in fact happens with zero probability), N is clearly right-continuous at t . Otherwise, by definition $N_s = N_t$ for all $s \in [t, \tau_1 + \dots + \tau_{N_t+1}]$ (this interval is non-empty by the definition of N_t).

7) We induct on $j-i$. The case $j-i = 0$ is clear by the definition. For the induction step, note that $\tau_{[i,j]} = \tau_{[i,j-1]} + \tau_j$ for $j-i \geq 1$ and that the two terms in the sum are independent. From the induction hypothesis we know the law of the two terms and (by convolution of the densities) we obtain that the corresponding density is

$$\int_{\mathbb{R}} \frac{s^{j-i-1} e^{-s}}{(j-i-1)!} \mathbf{1}_{(0,\infty)}(s) e^{-(t-s)} \mathbf{1}_{(0,\infty)}(t-s) ds = e^{-t} \mathbf{1}_{(0,\infty)}(t) \int_0^t \frac{s^{j-i-1}}{(j-i-1)!} ds = \frac{t^{j-i}}{(j-i)!} e^{-t} \mathbf{1}_{(0,\infty)}(t).$$

8) Fix $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \dots < t_n$. Then for $0 = k_0 \leq k_1 \leq \dots \leq k_n$,

$$\begin{aligned} \mathbb{P}[N_{t_i} = k_i, \text{ for } i \in \{1, \dots, n\}] &= \mathbb{P}[\tau_{[1,i]} \in (t_{j-1}, t_j], \text{ for } i \in \{k_{j-1} + 1, \dots, k_j\} \text{ and } j \in \{1, \dots, n\}, \tau_{[1, k_n+1]} > t_n] \\ &= \mathbb{P}[\tau_{[1,i]} \in (t_{j-1}, t_j] \text{ for } i \in \{k_{j-1} + 1, \dots, k_j\} \text{ and } j \in \{1, \dots, n-1\}, A_n], \end{aligned}$$

where A_n is the event (this is best seen by drawing a timeline for the jumps of the process)

$$A_n := \left\{ \tau_{k_{n-1}+1} > t_{n-1} - \tau_{[1, k_{n-1}]}, \tau_{[k_{n-1}+1, k_n]} - (t_{n-1} - \tau_{[1, k_{n-1}]}) \in (0, t_n - t_{n-1}), \right. \\ \left. \tau_{k_n+1} > t_n - t_{n-1} - (\tau_{[k_{n-1}+1, k_n]} - (t_{n-1} - \tau_{[1, k_{n-1}]}) \right\}.$$

We now condition on $\tau_1, \dots, \tau_{k_{n-1}}$ and observe that by the memoryless property of the exponential distribution, conditionally on $\tau_{k_{n-1}+1} > t_{n-1} - \tau_{[1, k_{n-1}]}$, $\tilde{\tau} = \tau_{k_{n-1}+1} - (t_{n-1} - \tau_{[1, k_{n-1}]})$ is exponentially distributed and by the previous questions

$$\tau_{[k_{n-1}+1, k_n]} - (t_{n-1} - \tau_{[1, k_{n-1}]}) = \tilde{\tau} + \sum_{k_{n-1}+1 < i \leq k_n} \tau_i$$

(conditioned on $\tilde{\tau} > 0$) is therefore distributed according to a Gamma distribution with shape parameter $k_n - k_{n-1}$ and rate 1. Therefore whenever $k_n - k_{n-1} \geq 1$, we get

$$\begin{aligned} \mathbb{P}[A_n | \tau_1, \dots, \tau_{k_{n-1}}] &= e^{-(t_{n-1} - \tau_{[1, k_{n-1}]})} \int_0^{t_n - t_{n-1}} \frac{e^{-t} t^{k_n - k_{n-1} - 1}}{(k_n - k_{n-1} - 1)!} e^{-(t_n - t_{n-1} - t)} dt \\ &= e^{-(t_{n-1} - \tau_{[1, k_{n-1}]})} \cdot \frac{e^{-(t_n - t_{n-1})} (t_n - t_{n-1})^{k_n - k_{n-1}}}{(k_n - k_{n-1})!} \\ &= \mathbb{P}[\tau_{[1, k_{n-1}+1]} > t_{n-1} | \tau_1, \dots, \tau_{k_{n-1}}] \frac{e^{-(t_n - t_{n-1})} (t_n - t_{n-1})^{k_n - k_{n-1}}}{(k_n - k_{n-1})!}, \end{aligned}$$

and one obtains the same result for $k_n - k_{n-1} = 0$ using a similar argument. Combining everything finally yields

$$\mathbb{P}[N_{t_i} = k_i, \text{ for } i \in \{1, \dots, n\}] = \mathbb{P}[N_{t_i} = k_i, \text{ for } i \in \{1, \dots, n-1\}] \frac{e^{-(t_n - t_{n-1})} (t_n - t_{n-1})^{k_n - k_{n-1}}}{(k_n - k_{n-1})!}.$$

Since \mathbb{P} -a.s., $N_{t_i} \in \mathbb{N}$ for all $i \in \{1, \dots, n\}$, this uniquely characterises the law of the vector $(N_{t_i})_{i \in \{1, \dots, n\}}$ and we see that N is indeed a Poisson process (of rate 1).

Exercise 2

We now use the Poisson process to construct some more complicated processes with independent stationary increments, that jump at a random dense set of times. Let $(N^{(n)})_{n \in \mathbb{Z}}$ be a sequence of i.i.d. Poisson processes and define

$$Y_t := \sum_{n=0}^{\infty} 4^{-n} N_{3^n t}^{(n)}, \text{ and } Z_t := \sum_{n \in \mathbb{Z}} 4^{-n} N_{3^n t}^{(n)}, t \geq 0.$$

Answer the questions below.

- 1) Compute $\mathbb{E}^{\mathbb{P}}[Y_t]$ for $t \geq 0$. Show that $Y_t < +\infty$, \mathbb{P} -a.s. for $t \geq 0$ and that Y has \mathbb{P} -independent and stationary increments.
- 2) Fix $t \geq 0$. Show that \mathbb{P} -a.s., Y is continuous at t .
- 3) Show that \mathbb{P} -almost surely, for all intervals $(a, b) \subset [0, +\infty)$, Y is not continuous on (a, b) . Show that \mathbb{P} -a.s., Y is increasing on $[0, +\infty)$.
- 4) What can you say about $\mathbb{E}^{\mathbb{P}}[Z_t]$?
- 5) Fix $T > 0$. Show that \mathbb{P} -a.s., there exists $n_0 \in \mathbb{N}$ such that $N_{3^{-n}T}^{(-n)} = 0$ for all $n \geq n_0$. Deduce that the sum in the definition of Z_t converges \mathbb{P} -a.s.
- 6) Show that Z and $(4Z_{t/3}; t \geq 0)$ have the same law.

The processes Y and Z we constructed above are examples of non-trivial subordinators (*i.e.* non-decreasing Lévy processes).

1) By monotone convergence, for $t \geq 0$

$$\mathbb{E}^{\mathbb{P}}[Y_t] = \sum_{n=0}^{\infty} 4^{-n} \mathbb{E}^{\mathbb{P}}[N_{3^n t}^{(n)}] = \sum_{n=0}^{\infty} (3/4)^n t = 4t.$$

Hence $\mathbb{E}^{\mathbb{P}}[Y_t] < +\infty$ and in particular, $Y_t < \infty$, \mathbb{P} -a.s. Since $N^{(n)}$ has stationary and independent increments, so does the process $(4^{-n} N_{3^n t}^{(n)})_{t \geq 0}$, and hence so does the sum of these independent terms.

2) Y is non-decreasing, so left-continuity at t is equivalent to $Y_{t-t2^{-n}} \rightarrow Y_t$ as $n \rightarrow \infty$. We can compute using $Y_{t-t2^{-n}} \leq Y_t$ and stationarity

$$\mathbb{E}^{\mathbb{P}} \left[\sum_{n \in \mathbb{N}} |Y_t - Y_{t-t2^{-n}}| \right] = \sum_{n \in \mathbb{N}} \mathbb{E}^{\mathbb{P}}[Y_t - Y_{t-t2^{-n}}] = \sum_{n \in \mathbb{N}} \mathbb{E}^{\mathbb{P}}[Y_{t2^{-n}}] = \sum_{n \in \mathbb{N}} 4t2^{-n} < \infty.$$

So the sum in the expectation \mathbb{P} -a.s. converges and in particular, $Y_{t-t2^{-n}} \rightarrow Y_t$, \mathbb{P} - a.s. Right-continuity is proved completely analogously.

3) Let $A_{(a,b)}$ be the event that Y is continuous on (a,b) . Then clearly

$$\cup_{0 \leq a < b} A_{(a,b)} = \cup_{\{(p,q) \in \mathbb{Q}^2: 0 \leq p < q\}} A_{(p,q)}.$$

So it is enough to show (since the union on the right is countable) that $\mathbb{P}[A_{(p,q)}] = 0$ for rational $0 \leq p < q$. Observe that

$$A_{(p,q)} = \{Y_{q-} = Y_p\} = \cap_{n \in \mathbb{N}} \{N_{3^n q-}^{(n)} - N_{3^n p}^{(n)} = 0\},$$

since Y is non-decreasing. So $\mathbb{P}[A_{(p,q)}] \leq \mathbb{P}[N_{3^n q-}^{(n)} - N_{3^n p}^{(n)} = 0] = e^{-3^n(q-p)} \rightarrow 0$ as $n \rightarrow \infty$, as required. The fact, that Y is \mathbb{P} -a.s. increasing is also immediate from this.

4) As in 1), $\mathbb{E}^{\mathbb{P}}[Z_t] = \sum_{n \in \mathbb{Z}} (3/4)^n t$ which is ∞ for $t > 0$ and 0 for $t = 0$.

5) For the first claim, observe that

$$\sum_{n \geq 0} \mathbb{P}[N_{3^{-n}T}^{(-n)} > 0] = \sum_{n \geq 0} (1 - e^{-3^{-n}T}) \leq \sum_{n \geq 0} 3^{-n}T < \infty.$$

So by Borel–Cantelli’s lemma, \mathbb{P} -a.s., $N_{3^{-n}T}^{(-n)} > 0$ occurs for only finitely many $n \geq 0$ as required. The final claim is obvious as the defining sum of Y_t and Z_t only differ in finitely many terms (with negative index).

6) We have

$$4Z_{t/3} = 4 \sum_{n \in \mathbb{Z}} 4^{-n} N_{3^n t/3}^{(n)} = \sum_{n \in \mathbb{Z}} 4^{-(n-1)} N_{3^{(n-1)}t}^{(n)} = \sum_{n \in \mathbb{Z}} 4^{-n} N_{3^n t}^{(n+1)}.$$

Since $N^{(n)}$ and $N^{(n+1)}$ have the same law, the claim follows.